

DOUBLING CONSTRUCTION OF CALABI-YAU THREEFOLDS

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Abstract. We give a differential-geometric construction and examples of Calabi-Yau threefolds, at least one out of which is *new*. Ingredients in our construction are *admissible pairs* introduced by Kovalev and Lee in [13], where an admissible pair (\bar{X}, D) consists of a three-dimensional compact Kähler manifold \bar{X} and a smooth anticanonical $K3$ divisor D on \bar{X} . If two admissible pairs (\bar{X}_1, D_1) and (\bar{X}_2, D_2) satisfy the *gluing condition*, we can glue $\bar{X}_1 \setminus D_1$ and $\bar{X}_2 \setminus D_2$ together to obtain a Calabi-Yau threefold M . In particular, if (\bar{X}_1, D_1) and (\bar{X}_2, D_2) are identical to an admissible pair (\bar{X}, D) , then the gluing condition holds automatically, so that we can *always* construct a Calabi-Yau threefold from a single admissible pair (\bar{X}, D) by *doubling* it. Furthermore, we can compute all Betti and Hodge numbers of the resulting Calabi-Yau threefolds in the doubling construction.

1. INTRODUCTION

The purpose of this paper is to give a gluing construction and examples of Calabi-Yau threefolds. Before going into the details, we recall the some historical background of our gluing construction.

The gluing technique is used in constructing many compact manifolds with a special geometric structure. In particular, it is effectively used in constructing compact manifolds with exceptional holonomy groups G_2 and $\text{Spin}(7)$, which are also called compact G_2 - and $\text{Spin}(7)$ -manifolds respectively. The first examples of compact G_2 - and $\text{Spin}(7)$ - manifolds were obtained by Joyce using Kummer-type constructions in a series of his papers [8, 9, 10]. Also, Joyce gave a second construction of compact $\text{Spin}(7)$ -manifolds using compact four-dimensional Kähler orbifolds with an antiholomorphic involution. These constructions are based on the resolution of certain singularities by replacing neighborhoods of singularities with ALE-type manifolds. Later, Clancy studied in [2] such compact Kähler orbifolds systematically and constructed more new examples of compact $\text{Spin}(7)$ -manifolds using Joyce's second construction.

On the other hand, Kovalev gave another construction of compact G_2 -manifolds in [12]. Beginning with a Fano threefold \bar{W} with a smooth anticanonical $K3$ divisor D , he showed that if we blow up \bar{W} along a curve representing $D \cdot D$ to obtain \bar{X} , then \bar{X} has an anticanonical divisor isomorphic to D (denoted by D again) with the holomorphic normal bundle $N_{D/\bar{X}}$ trivial. Then $\bar{X} \setminus D$ admits an asymptotically cylindrical Ricci-flat Kähler metric. (Later Kovalev and Lee called such (\bar{X}, D) as an *admissible pair of Fano type*.) Also, Kovalev proved that if two admissible pairs (\bar{X}_1, D_1) and (\bar{X}_2, D_2) satisfy a certain condition called the *matching condition*, we can glue together $(X_1 \setminus D_1) \times S^1$ and $(X_2 \setminus D_2) \times S^1$ along their cylindrical ends in a *twisted* manner to obtain a compact G_2 -manifold. In this construction Kovalev found many new examples of G_2 -manifolds using the classification of Fano threefolds by Mori and Mukai [14, 15]. Later, Kovalev and Lee [13] found admissible pairs of another type (called *admissible pairs of non-symplectic type*) and constructed new examples of compact G_2 -manifolds. They used the classification of $K3$ surfaces with a non-symplectic involution by Nikulin [17].

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In our construction, we begin with two admissible pairs (\bar{X}_1, D_1) and (\bar{X}_2, D_2) as above. Then each $(\bar{X}_i \setminus D_i) \times S^1$ has a natural asymptotically cylindrical torsion-free G_2 -structure $\varphi_{i,\text{cyl}}$ using the existence result of an asymptotically cylindrical Ricci-flat Kähler form on $\bar{X}_i \setminus D_i$. Now suppose $\bar{X}_1 \setminus D_1$ and $\bar{X}_2 \setminus D_2$ have the same *asymptotic model*, which is ensured by the *gluing condition* defined later. Then as in Kovalev's construction, we can glue together $(\bar{X}_1 \setminus D_1) \times S^1$ and $(\bar{X}_2 \setminus D_2) \times S^1$, but in a *non-twisted* manner to obtain $M_T \times S^1$. In short, we glue together $(\bar{X}_1 \setminus D_1)$ and $(\bar{X}_2 \setminus D_2)$ along their cylindrical ends $D_1 \times S^1 \times (T-1, T+1)$ and $D_2 \times S^1 \times (T-1, T+1)$, and then take the product with S^1 . Moreover, we can glue together torsion-free G_2 -structures to construct a d-closed G_2 -structure φ_T on $M_T \times S^1$. Using the analysis on torsion-free G_2 -structures, we shall prove that φ_T can be deformed into a torsion-free G_2 -structure for sufficiently large T , so that the resulting compact manifold $M_T \times S^1$ admits a Riemannian metric with holonomy contained in G_2 . But if $M = M_T$ is simply-connected, then M must have holonomy $\text{SU}(3)$ according to the Berger-Simons classification of holonomy groups of Ricci-flat Riemannian manifolds. Hence this M is a Calabi-Yau threefold.

For two given admissible pairs (\bar{X}_1, D_1) and (\bar{X}_2, D_2) , it is difficult to check whether the gluing condition holds in general. However, if (\bar{X}_1, D_1) and (\bar{X}_2, D_2) are identical to an admissible pair (\bar{X}, D) , then the gluing condition holds automatically. Therefore we can *always* construct a Calabi-Yau threefold from a single admissible pair (\bar{X}, D) by *doubling* it.

Our doubling construction has another advantage in computing Betti and Hodge numbers of the resulting Calabi-Yau threefolds M . To compute Betti numbers of M , it is necessary to find out the intersection of the images of the homomorphisms $H^2(X_i, \mathbb{R}) \rightarrow H^2(D_i, \mathbb{R})$ for $i = 1, 2$ induced by the inclusion $D_i \times S^1 \rightarrow X_i$, where we denote $X_i = \bar{X}_i \setminus D_i$. In the doubling construction, the above two homomorphisms are identical, and the intersection of their images is the same as each one.

With this construction, we shall give 123 topologically distinct Calabi-Yau threefolds (59 examples from admissible pairs of Fano type and 64 from those of non-symplectic type). Out of 123 examples, at least one example is new, which arises from admissible pairs of non-symplectic type.

This paper is organized as follows. Section 2 is a brief review of G_2 -structures. In Section 3 we establish our gluing construction of Calabi-Yau threefolds from admissible pairs. The rest of the paper is devoted to constructing examples and compute Betti and Hodge numbers of Calabi-Yau threefolds obtained in our doubling construction. The reader who is not familiar with analysis can check Definition 3.6 of admissible pairs, go to Section 3.4 where the gluing theorems are stated, and then proceed to Section 4, skipping Section 2 and the rest of Section 3. In Section 4 we will find a formula for computing Betti numbers of the resulting Calabi-Yau threefolds M in our doubling construction. In Section 5, we recall two types of admissible pairs and rewrite the formula given in Section 4.1 to obtain formulas of Betti and Hodge numbers of M in terms of certain invariants which characterize admissible pairs. Then the last section lists all data of the Calabi-Yau threefolds obtained in our construction.

The first author is mainly responsible for Sections 1–3, and the second author mainly for Sections 4–6.

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2. GEOMETRY OF G_2 -STRUCTURES

Here we shall recall some basic facts about G_2 -structures on oriented 7-manifolds. For more details, see [11].

We begin with the definition of G_2 -structures on oriented vector spaces of dimension 7.

Definition 2.1. Let V be an oriented real vector space of dimension 7. Let $\{\theta^1, \dots, \theta^7\}$ be an oriented basis of V . Set

$$(2.1) \quad \begin{aligned} \varphi_0 &= \theta^{123} + \theta^{145} + \theta^{167} + \theta^{246} - \theta^{257} - \theta^{347} - \theta^{356}, \\ g_0 &= \sum_{i=1}^7 \theta^i \otimes \theta^i, \end{aligned}$$

where $\theta^{ij\dots k} = \theta^i \wedge \theta^j \wedge \dots \wedge \theta^k$. Define the $\mathrm{GL}_+(V)$ -orbit spaces

$$\begin{aligned} \mathcal{P}^3(V) &= \{ a^* \varphi_0 \mid a \in \mathrm{GL}_+(V) \}, \\ \mathcal{Met}(V) &= \{ a^* g_0 \mid a \in \mathrm{GL}_+(V) \}. \end{aligned}$$

We call $\mathcal{P}^3(V)$ the set of *positive 3-forms* (also called the set of G_2 -structures or *associative 3-forms*) on V . On the other hand, $\mathcal{Met}(V)$ is the set of positive-definite inner products on V , which is also a homogeneous space isomorphic to $\mathrm{GL}_+(V)/\mathrm{SO}(V)$, where $\mathrm{SO}(V)$ is defined by

$$\mathrm{SO}(V) = \{ a \in \mathrm{GL}_+(V) \mid a^* g_0 = g_0 \}.$$

Now the group G_2 is defined as the isotropy of the action of $\mathrm{GL}(V)$ (in place of $\mathrm{GL}_+(V)$) on $\mathcal{P}^3(V)$ at φ_0 :

$$G_2 = \{ a \in \mathrm{GL}(V) \mid a^* \varphi_0 = \varphi_0 \}.$$

Then one can show that G_2 is a compact Lie group of dimension 14 which is a Lie subgroup of $\mathrm{SO}(V)$ [5]. Thus we have a natural projection

$$(2.2) \quad \mathcal{P}^3(V) \cong \mathrm{GL}_+(V)/G_2 \longrightarrow \mathrm{GL}_+(V)/\mathrm{SO}(V) \cong \mathcal{Met}(V),$$

so that each positive 3-form (or G_2 -structure) $\varphi_0 \in \mathcal{P}^3(V)$ defines a positive-definite inner product $g_{\varphi_0} \in \mathcal{Met}(V)$ on V . Note that both $\mathcal{P}^3(V)$ and $\mathcal{Met}(V)$ depend only on the orientation of V and are independent of the choice of an oriented basis $\{\theta^1, \dots, \theta^7\}$, and so is the map (2.2). Note also that

$$\dim_{\mathbb{R}} \mathcal{P}^3(V) = \dim_{\mathbb{R}} \mathrm{GL}_+(V) - \dim_{\mathbb{R}} G_2 = 7^2 - 14 = 35,$$

which is the same as $\dim_{\mathbb{R}} \wedge^3 V$. This implies that $\mathcal{P}^3(V)$ is an *open* subset of $\wedge^3 V$. The following lemma is immediate.

Lemma 2.2. *There exists a constant $\rho_* > 0$ such that for any $\varphi_0 \in \mathcal{P}^3(V)$, if $\tilde{\varphi}_0 \in \wedge^3 V$ satisfies $|\tilde{\varphi}_0 - \varphi_0|_{g_{\varphi_0}} < \rho_*$, then $\tilde{\varphi}_0 \in \mathcal{P}^3(V)$.*

Remark 2.3. Here is an alternative definition of G_2 -structures. But the reader can skip the following. Let V be an oriented real vector space of dimension 7 with orientation μ_0 . Let $\Omega_0 \in \wedge^7 V^*$ be a volume form which is positive with respect to the orientation μ_0 . Then $\varphi_0 \in \wedge^3 V^*$ is a positive 3-form on V if an inner product g_{Ω_0, φ_0} given by

$$(2.3) \quad \iota_u \varphi_0 \wedge \iota_v \varphi_0 \wedge \varphi_0 = 6 g_{\Omega_0, \varphi_0}(u, v) \Omega_0 \quad \text{for } u, v \in V$$

is positive-definite, where ι_u denotes interior product by a vector $u \in V$, from which comes the name ‘positive form’. Whether φ_0 is a positive 3-form depends only on the orientation μ_0 of V , and is independent of the choice of a positive volume form Ω_0 . One can show that if φ_0 is a positive 3-form on (V, μ_0) , then there exists a unique positive-definite inner product g_{φ_0} such that

$$(2.4) \quad \iota_u \varphi_0 \wedge \iota_v \varphi_0 \wedge \varphi_0 = 6 g_{\varphi_0}(u, v) \mathrm{vol}_{g_{\varphi_0}} \quad \text{for } u, v \in V,$$

where vol_{φ_0} is a volume form determined by g_{φ_0} and μ_0 . The map $\varphi_0 \mapsto g_{\varphi_0}$ gives (2.2) explicitly. One can also prove that there exists an orthonormal basis $\{\theta^1, \dots, \theta^7\}$ with respect to g_{φ_0} such that φ_0 and g_{φ_0} are written in the form of (2.1).

Now we define G_2 -structures on oriented 7-manifolds.

Definition 2.4. Let M be an oriented 7-manifold. We define $\mathcal{P}^3(M) \rightarrow M$ to be the fiber bundle whose fiber over x is $\mathcal{P}^3(T_x^*M) \subset \wedge^3 T_x^*M$. Then $\varphi \in C^\infty(\wedge^3 T^*M)$ is a *positive 3-form* (also an *associative 3-form* or a G_2 -structure) on M if $\varphi \in C^\infty(\mathcal{P}^3(M))$, i.e., φ is a smooth section of $\mathcal{P}^3(M)$. If φ is a G_2 -structure on M , then φ induces a Riemannian metric g_φ since each $\varphi|_x$ for $x \in M$ induces a positive-definite inner product $g_{\varphi|_x}$ on $T_x M$. A G_2 -structure φ on M is said to be *torsion-free* if it is parallel with respect to the induced Riemannian metric g_φ , i.e., $\nabla_{g_\varphi} \varphi = 0$, where ∇_{g_φ} is the Levi-Civita connection of g_φ .

Lemma 2.5. Let ρ_* be the constant given in Lemma 2.2. For any $\varphi \in \mathcal{P}^3(M)$, if $\tilde{\varphi} \in C^\infty(\wedge^3 T^*M)$ satisfies $\|\tilde{\varphi} - \varphi\|_{C^0} < \rho_*$, then $\tilde{\varphi} \in \mathcal{P}^3(M)$, where $\|\cdot\|_{C^0}$ is measured using the metric g_φ on M .

The following result is one of the most important results in the geometry of the exceptional holonomy group G_2 , relating the holonomy contained in G_2 with the d- and d*-closedness of the G_2 -structure.

Theorem 2.6 (Salamon [20], Lemma 11.5). *Let M be an oriented 7-manifold. Let φ be a G_2 -structure on M and g_φ the induced Riemannian metric on M . Then the following conditions are equivalent.*

- (1) φ is a torsion-free G_2 -structure, i.e., $\nabla_{g_\varphi} \varphi = 0$.
- (2) $d\varphi = d *_{g_\varphi} \varphi = 0$, where $*_{g_\varphi}$ is the Hodge star operator induced by g_φ .
- (3) $d\varphi = d^*_{g_\varphi} \varphi = 0$, where $d^*_{g_\varphi} = - *_{g_\varphi} d *_{g_\varphi}$ is the formal adjoint operator of d .
- (4) The holonomy group $\text{Hol}(g_\varphi)$ of g_φ is contained in G_2 .

3. THE GLUING PROCEDURE

3.1. Compact complex manifolds with an anticanonical divisor. We suppose that \overline{X} is a compact complex manifold of dimension m , and D is a smooth irreducible anticanonical divisor on \overline{X} . We recall some results in [4], Sections 3.1 and 3.2.

Lemma 3.1. *Let \overline{X} be a compact complex manifold of dimension m and D a smooth irreducible anticanonical divisor on \overline{X} . Then there exists a local coordinate system $\{U_\alpha, (z_\alpha^1, \dots, z_\alpha^{m-1}, w_\alpha)\}$ on \overline{X} such that*

- (i) w_α is a local defining function of D on U_α , i.e., $D \cap U_\alpha = \{w_\alpha = 0\}$, and
- (ii) the m -forms $\Omega_\alpha = \frac{dw_\alpha}{w_\alpha} \wedge dz_\alpha^1 \wedge \dots \wedge dz_\alpha^{m-1}$ on U_α together yield a holomorphic volume form Ω on $X = \overline{X} \setminus D$.

Next we shall see that $X = \overline{X} \setminus D$ is a cylindrical manifold whose structure is induced from the holomorphic normal bundle $N = N_{D/\overline{X}}$ to D in \overline{X} , where the definition of cylindrical manifolds is given as follows.

Definition 3.2. Let X be a noncompact differentiable manifold of dimension n . Then X is called a *cylindrical manifold* or a *manifold with a cylindrical end* if there exists a diffeomorphism $\pi : X \setminus X_0 \rightarrow \Sigma \times \mathbb{R}_+ = \{(p, t) \mid p \in \Sigma, 0 < t < \infty\}$ for some compact submanifold X_0 of dimension n with boundary $\Sigma = \partial X_0$. Also, extending t to X so that $t \leq 0$ on $X \setminus X_0$, we call t a *cylindrical parameter* on X .

Let (x_α, y_α) be local coordinates on $V_\alpha = U_\alpha \cap D$, such that x_α is the restriction of z_α to V_α and y_α is a coordinate in the fiber direction. Then one can see easily that $dx_\alpha^1 \wedge \dots \wedge dx_\alpha^{m-1}$ on V_α

together yield a holomorphic volume form Ω_D , which is also called the *Poincaré residue* of Ω along D . Let $\|\cdot\|$ be the norm of a Hermitian bundle metric on N . We can define a cylindrical parameter t on N by $t = -\frac{1}{2} \log \|s\|^2$ for $s \in N \setminus D$. Then the local coordinates (z_α, w_α) on X are asymptotic to the local coordinates (x_α, y_α) on $N \setminus D$ in the following sense.

Lemma 3.3. *There exists a diffeomorphism Φ from a neighborhood V of the zero section of N containing $t^{-1}(\mathbb{R}_+)$ to a tubular neighborhood of U of D in X such that Φ can be locally written as*

$$(3.1) \quad \begin{aligned} z_\alpha &= x_\alpha + O(|y_\alpha|^2) = x_\alpha + O(e^{-t}), \\ w_\alpha &= y_\alpha + O(|y_\alpha|^2) = y_\alpha + O(e^{-t}), \end{aligned}$$

where we multiply all z_α and w_α by a single constant to ensure $t^{-1}(\mathbb{R}_+) \subset V$ if necessary.

Hence X is a cylindrical manifold with the cylindrical parameter t via the diffeomorphism Φ given in the above lemma. In particular, when $H^0(\bar{X}, \mathcal{O}_{\bar{X}}) = 0$ and $N_{D/\bar{X}}$ is trivial, we have a useful coordinate system near D .

Lemma 3.4. *Let (\bar{X}, D) be as in Lemma 3.1. If $H^1(\bar{X}, \mathcal{O}_{\bar{X}}) = 0$ and the normal bundle $N_{D/\bar{X}}$ is holomorphically trivial, then there exists an open neighborhood U_D of D and a holomorphic function w on U_D such that w is a local defining function of D on U_D . Also, we may define the cylindrical parameter t with $t^{-1}(\mathbb{R}_+) \subset U_D$ by writing the fiber coordinate y of $N_{D/\bar{X}}$ as $y = \exp(-t - \sqrt{-1}\theta)$.*

Proof. We deduce from the short exact sequence

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{O}_X & \longrightarrow & [D] & \longrightarrow & [D]|_D \longrightarrow 0 \\ & & & & & & \parallel \\ & & & & & & N_{D/\bar{X}} \cong \mathcal{O}_D \end{array}$$

the long exact sequence

$$\begin{array}{ccccccc} \cdots & \longrightarrow & H^0(X, [D]) & \longrightarrow & H^0(D, N_{D/\bar{X}}) & \longrightarrow & H^1(\bar{X}, \mathcal{O}_{\bar{X}}) \longrightarrow \cdots \\ & & & & \parallel & & \parallel \\ & & & & H^0(D, \mathcal{O}_D) \cong \mathbb{C} & & 0 \end{array}$$

Thus there exists a holomorphic section $s \in H^0(X, [D])$ such that $s|_D \equiv 1 \in H^0(D, N_{D/\bar{X}})$. Setting $U_D = \{x \in X \mid s(x) \neq 0\}$, we have $[D]|_{U_D} \cong \mathcal{O}_{U_D}$, so that there exists a local defining function w of D on U_D . \square

3.2. Admissible pairs and asymptotically cylindrical Ricci-flat Kähler manifolds.

Definition 3.5. Let X be a cylindrical manifold such that $\pi : X \setminus X_0 \rightarrow \Sigma \times \mathbb{R}_+ = \{(p, t)\}$ is a corresponding diffeomorphism. If g_Σ is a Riemannian metric on Σ , then it defines a cylindrical metric $g_{\text{cyl}} = g_\Sigma + dt^2$ on $\Sigma \times \mathbb{R}_+$. Then a complete Riemannian metric g on X is said to be *asymptotically cylindrical* (to $(\Sigma \times \mathbb{R}_+, g_{\text{cyl}})$) if g satisfies

$$\left| \nabla_{g_{\text{cyl}}}^j (g - g_{\text{cyl}}) \right|_{g_{\text{cyl}}} \longrightarrow 0 \quad \text{as } t \longrightarrow \infty \quad \text{for all } j \geq 0$$

for some cylindrical metric $g_{\text{cyl}} = g_\Sigma + dt^2$, where we regarded g_{cyl} as a Riemannian metric on $X \setminus X_0$ via the diffeomorphism π . Also, we call (X, g) an *asymptotically cylindrical manifold* and $(\Sigma \times \mathbb{R}_+, g_{\text{cyl}})$ the *asymptotic model* of (X, g) .

Definition 3.6. Let \bar{X} be a compact Kähler manifold and D a divisor on \bar{X} . Then (\bar{X}, D) is said to be an *admissible pair* if the following conditions hold:

- (a) \bar{X} is a compact Kähler manifold,

- (b) D is a smooth anticanonical divisor on \overline{X} ,
- (c) the normal bundle $N_{\overline{X}/D}$ is trivial, and
- (d) \overline{X} and $X = \overline{X} \setminus D$ are simply-connected.

From the above conditions, we see that Lemmas 3.1 and 3.4 apply for admissible pairs. Also, if $\dim_{\mathbb{C}} X = 3$, then D must be a $K3$ surface, i.e., a simply-connected compact complex surface with trivial canonical bundle.

Theorem 3.7 (Tian-Yau [21], Kovalev [12], Hein [6]). *Let (\overline{X}, ω') be a compact Kähler manifold and $m = \dim_{\mathbb{C}} \overline{X}$. If (\overline{X}, D) is an admissible pair, then the following is true.*

It follows from Lemmas 3.1 and 3.4, there exist a local coordinate system $(U_{D,\alpha}, (z_{\alpha}^1, \dots, z_{\alpha}^{m-1}, w))$ on a neighborhood $U_D = \cup_{\alpha} U_{D,\alpha}$ of D and a holomorphic volume form Ω on \overline{X} such that

$$(3.2) \quad \Omega = \frac{dw}{w} \wedge dz_{\alpha}^1 \wedge \dots \wedge dz_{\alpha}^{m-1} \quad \text{on } U_{D,\alpha}.$$

Let κ_D be the unique Ricci-flat Kähler form on D in the Kähler class $[\omega'|_D]$. Also let (x_{α}, y) be local coordinates of $N_{D/\overline{X}} \setminus D$ as in Section 3.1 and write y as $y = \exp(-t - \sqrt{-1}\theta)$. Now define a holomorphic volume form Ω_{cyl} and a cylindrical Ricci-flat Kähler form ω_{cyl} by

$$(3.3) \quad \begin{aligned} \Omega_{\text{cyl}} &= \frac{dy}{y} \wedge dx_{\alpha}^1 \wedge \dots \wedge dx_{\alpha}^{m-1} = (dt + \sqrt{-1}d\theta) \wedge \Omega_D, \\ \omega_{\text{cyl}} &= \kappa_D + \frac{dy \wedge d\bar{y}}{|y|^2} = \kappa_D + dt \wedge d\theta. \end{aligned}$$

Then there exists an asymptotically cylindrical Ricci-flat Kähler form ω on $X = \overline{X} \setminus D$ such that

$$\Omega - \Omega_{\text{cyl}} = d\zeta, \quad \omega - \omega_{\text{cyl}} = d\xi \quad \text{for some } \zeta \text{ and } \xi \text{ with}$$

$$\left| \nabla_{g_{\text{cyl}}}^j \zeta \right|_{g_{\text{cyl}}} = O(e^{-\beta t}), \quad \left| \nabla_{g_{\text{cyl}}}^j \xi \right|_{g_{\text{cyl}}} = O(e^{-\beta t}) \quad \text{for all } j \geq 0 \text{ and } 0 < \beta < \min \{ 1/2, \sqrt{\lambda_1} \},$$

where λ_1 is the first eigenvalue of the Laplacian $\Delta_{g_D + d\theta^2}$ acting on $D \times S^1$ with g_D the metric associated with κ_D .

A pair of a holomorphic volume form Ω and a Ricci-flat Kähler form ω on an m -dimensional Kähler manifold normalized so that

$$(3.4) \quad \frac{\omega^m}{m!} = \frac{(\sqrt{-1})^{m^2}}{2^m} \Omega \wedge \overline{\Omega} \quad (= \text{the volume form})$$

is called a *Calabi-Yau structure*. The above theorem states that there exists a Calabi-Yau structure (Ω, ω) on X asymptotic to a cylindrical Calabi-Yau structure $(\Omega_{\text{cyl}}, \omega_{\text{cyl}})$ on $N_{D/\overline{X}} \setminus D$ if we multiply Ω by some constant.

3.3. Gluing admissible pairs. Hereafter we only consider admissible pairs (\overline{X}, D) with $\dim_{\mathbb{C}} \overline{X} = 3$. Also, we denote $N = N_{D/\overline{X}}$ and $X = \overline{X} \setminus D$.

3.3.1. The gluing condition. Let (\overline{X}, ω') be a 3-dimensional compact Kähler manifold and (\overline{X}, D) be an admissible pair. We first define a natural torsion-free G_2 -structure on $X \times S^1$.

It follows from Theorem 3.7 that there exists a Calabi-Yau structure (Ω, ω) on X asymptotic to a cylindrical Calabi-Yau structure $(\Omega_{\text{cyl}}, \omega_{\text{cyl}})$ on $N \setminus D$, which are written as (3.2) and (3.3). We define a G_2 -structure φ on $X \times S^1$ by

$$(3.5) \quad \varphi = \omega \wedge d\theta' + \text{Im } \Omega,$$

where $\theta' \in \mathbb{R}/2\pi\mathbb{Z}$ is a coordinate on S^1 . Similarly, we define a G_2 -structure φ_{cyl} on $(N \setminus D) \times S^1$ by

$$(3.6) \quad \varphi_{\text{cyl}} = \omega_{\text{cyl}} \wedge d\theta' + \text{Im } \Omega_{\text{cyl}}.$$

Also, the Hodge duals of φ and φ_{cyl} are computed as

$$(3.7) \quad \begin{aligned} *_{g_\varphi} \varphi &= \frac{1}{2} \omega \wedge \omega - \text{Re } \Omega \wedge d\theta', \\ *_{g_{\varphi_{\text{cyl}}}} \varphi_{\text{cyl}} &= \frac{1}{2} \omega_{\text{cyl}} \wedge \omega_{\text{cyl}} - \text{Re } \Omega_{\text{cyl}} \wedge d\theta'. \end{aligned}$$

Then we see easily from Theorem 3.7 and equations (3.5)–(3.7) that

$$(3.8) \quad \begin{aligned} \varphi - \varphi_{\text{cyl}} &= d\xi \wedge d\theta' + \text{Im } d\zeta = d\eta_1, \\ *_{g_\varphi} \varphi - *_{g_{\varphi_{\text{cyl}}}} \varphi_{\text{cyl}} &= (\omega + \omega_{\text{cyl}}) \wedge d\xi - \text{Re } d\zeta \wedge d\theta' = d\eta_2, \\ \text{where } \eta_1 &= \xi \wedge d\theta' + \text{Im } \zeta, \quad \eta_2 = (\omega + \omega_{\text{cyl}}) \wedge \xi - \text{Re } \zeta \wedge d\theta'. \end{aligned}$$

Thus φ and φ_{cyl} are both torsion-free G_2 -structures, and $(X \times S^1, \varphi)$ is asymptotic to $((N \setminus D) \times S^1, \varphi_{\text{cyl}})$. Note that the cylindrical end of $X \times S^1$ is diffeomorphic to $(N \setminus D) \times S^1 \simeq D \times S^1 \times S^1 \times \mathbb{R}_+ = \{(x_\alpha, \theta, \theta', t)\}$.

Next we consider the condition under which we can glue together X_1 and X_2 obtained from admissible pairs (\bar{X}_1, D_1) and (\bar{X}_2, D_2) . For gluing X_1 and X_2 to obtain a manifold with an approximating G_2 -structure, we would like (X_1, φ_1) and (X_2, φ_2) to have the same asymptotic model. Thus we put the following

Gluing condition: There exists a diffeomorphism $F : D_1 \times S^1 \times S^1 \longrightarrow D_2 \times S^1 \times S^1$ between the cross-sections of the cylindrical ends such that

$$(3.9) \quad F_T^* \varphi_{2, \text{cyl}} = \varphi_{1, \text{cyl}} \quad \text{for all } T > 0,$$

where $F_T : D_1 \times S^1 \times S^1 \times (0, 2T) \longrightarrow D_2 \times S^1 \times S^1 \times (0, 2T)$ is defined by

$$F_T(x_1, \theta_1, \theta'_1, t) = (F(x_1, \theta_1, \theta'_1), 2T - t) \quad \text{for } (x_1, \theta_1, \theta'_1, t) \in D_1 \times S^1 \times S^1 \times (0, 2T).$$

Lemma 3.8. *Suppose that there exists an isomorphism $f : D_1 \longrightarrow D_2$ such that $f^* \kappa_{D_2} = \kappa_{D_1}$. If we define a diffeomorphism F between the cross-sections of the cylindrical ends by*

$$\begin{aligned} F_T : D_1 \times S^1 \times S^1 &\longrightarrow D_2 \times S^1 \times S^1. \\ \downarrow &\quad \quad \quad \downarrow \\ (x_1, \theta_1, \theta'_1) &\longmapsto (x_2, \theta_2, \theta'_2) = (f(x_1), -\theta_1, \theta'_1) \end{aligned}$$

Then the gluing condition (3.9) holds, where we change the sign of $\Omega_{2, \text{cyl}}$ (and also the sign of Ω_2 correspondingly).

Proof. It follows by a straightforward calculation using (3.3) and (3.6). \square

Remark 3.9. In the constructions of compact G_2 -manifolds by Kovalev[12] and Kovalev-Lee[13], the map $F : D_1 \times S^1 \times S^1 \longrightarrow D_2 \times S^1 \times S^1$ is defined by

$$F(x_1, \theta_1, \theta'_1) = (x_2, \theta_2, \theta'_2) = (f(x_1), \theta'_1, \theta_1) \quad \text{for } (x_1, \theta_1, \theta'_1) \in D_1 \times S^1 \times S^1,$$

so that F twists the two S^1 factors. Then in order for the gluing condition (3.9) to hold, the isomorphism $f : D_1 \longrightarrow D_2$ between $K3$ surfaces must satisfy

$$f^* \kappa_2^I = -\kappa_1^J, \quad f^* \kappa_2^J = \kappa_1^I, \quad f^* \kappa_2^K = \kappa_1^K,$$

where $\kappa_i^I, \kappa_i^J, \kappa_i^K$ are defined by

$$\kappa_{D_i} = \kappa_i^I, \quad \Omega_{D_i} = \kappa_i^J + \sqrt{-1} \kappa_i^K.$$

Instead, Kovalev and Lee put a weaker condition (which they call the *matching condition*)

$$f^*[\kappa_2^I] = -[\kappa_1^J], \quad f^*[\kappa_2^J] = [\kappa_1^I], \quad f^*[\kappa_2^K] = [\kappa_1^K],$$

which is sufficient for the existence of f by the global Torelli theorem of $K3$ surfaces.

3.3.2. *Approximating G_2 -structures.* Now we shall glue $X_1 \times S^1$ and $X_2 \times S^1$ under the gluing condition (3.9). Let $\rho : \mathbb{R} \rightarrow [0, 1]$ denote a cut-off function

$$\rho(x) = \begin{cases} 1 & \text{if } x \leq 0, \\ 0 & \text{if } x \geq 1, \end{cases}$$

and define $\rho_T : \mathbb{R} \rightarrow [0, 1]$ by

$$\rho_T(x) = \rho(x - T + 1) = \begin{cases} 1 & \text{if } x \leq T - 1, \\ 0 & \text{if } x \geq T. \end{cases}$$

Setting an approximating Calabi-Yau structure $(\Omega_{i,T}, \omega_{i,T})$ by

$$(3.10) \quad \Omega_{i,T} = \begin{cases} \Omega_i - d(1 - \rho_{T-1})\zeta_i & \text{on } \{t \leq T - 1\}, \\ \Omega_{i,\text{cyl}} + d\rho_{T-1}\zeta_i & \text{on } \{t \geq T - 2\} \end{cases}$$

and similarly

$$(3.11) \quad \omega_{i,T} = \begin{cases} \omega_i - d(1 - \rho_{T-1})\xi_i & \text{on } \{t \leq T - 1\}, \\ \omega_{i,\text{cyl}} + d\rho_{T-1}\xi_i & \text{on } \{t \geq T - 2\}, \end{cases}$$

we can define a d-closed (but not necessarily d*-closed) G_2 -structure $\varphi_{i,T}$ on each $X_i \times S^1$ by

$$(3.12) \quad \varphi_{i,T} = \omega_{i,T} \wedge d\theta'_i + \text{Im } \Omega_T.$$

Note that $\varphi_{i,T}$ satisfies

$$\varphi_{i,T} = \begin{cases} \varphi_i & \text{on } \{t < T - 2\}, \\ \varphi_{i,\text{cyl}} & \text{on } \{t > T - 1\} \end{cases}$$

and that

$$(3.13) \quad |\varphi_{i,T} - \varphi_{i,\text{cyl}}|_{g_{\varphi_{i,\text{cyl}}}} = O(e^{-\beta T}) \quad \text{for all } 0 < \beta < \min \{1/2, \sqrt{\lambda_1}\}.$$

Let $X_{1,T} = \{t_1 < T + 1\} \subset X_1$ and $X_{2,T} = \{t_2 < T + 1\} \subset X_2$. We glue $X_{1,T} \times S^1$ and $X_{2,T} \times S^1$ along $D_1 \times S^1 \times \{T - 1 < t_1 < T + 1\} \times S^1 \subset X_{1,T} \times S^1$ and $D_2 \times S^1 \times \{T - 1 < t_2 < T + 1\} \times S^1 \subset X_{2,T} \times S^1$ to construct a compact 7-manifold $M_T \times S^1$ using the gluing map F_T (more precisely, $\tilde{F}_T = (\Phi_2, \text{id}_{S^1}) \circ F_T \circ (\Phi_1^{-1}, \text{id}_{S^1})$, where Φ_1 and Φ_2 are the diffeomorphisms given in Lemma 3.3). Also, we can glue together $\varphi_{1,T}$ and $\varphi_{2,T}$ to obtain a 3-form φ_T on M_T . It follows from Lemma 2.5 and (3.13) that there exists $T_* > 0$ such that $\varphi_T \in \mathcal{P}^3(M_T \times S^1)$ for all T with $T > T_*$, so that the Hodge star operator $* = *_{g_{\varphi_T}}$ is well-defined. Thus we can define a 3-form ψ_T on $M_T \times S^1$ with $d^*\varphi_T = d^*\psi_T$ by

$$(3.14) \quad *\psi_T = *\varphi_T - \left(\frac{1}{2} \omega_T \wedge \omega_T - \text{Re } \Omega_T \wedge d\theta' \right).$$

Proposition 3.10. *There exist constants $A_\beta, B_{p,\beta}, C_{p,\beta}$ independent of T such that for β with $0 < \beta < \min \{1/2, \sqrt{\lambda_1}\}$ we have*

$$(3.15) \quad \|\psi_T\|_{C^0} \leq A_\beta e^{-\beta T}, \quad \|\psi_T\|_{L^p} \leq B_{p,\beta} e^{-\beta T}, \quad \|d^*\psi_T\|_{L^p} \leq C_{p,\beta} e^{-\beta T},$$

where all norms are measured using g_{φ_T} .

Proof. These estimates follow in a straightforward way from Theorem 3.7 and equation (3.8) by arguments similar to those in [4], Section 3.5. \square

3.4. Gluing construction of Calabi-Yau threefolds. Here we give the main theorems for constructing Calabi-Yau threefolds.

Theorem 3.11. *Let (\bar{X}_1, ω'_1) and (\bar{X}_2, ω'_2) be compact Kähler manifold with $\dim_{\mathbb{C}} \bar{X}_i = 3$ such that (\bar{X}_1, D_1) and (\bar{X}_2, D_2) are admissible pairs. Suppose there exists an isomorphism $f : D_1 \rightarrow D_2$ such that $f^* \kappa_2 = \kappa_1$, where κ_i is the unique Ricci-flat Kähler form on D_i in the Kähler class $[\omega'_i|_{D_i}]$. Then we can glue together X_1 and X_2 along their cylindrical ends to obtain a compact manifold M . The manifold M is a Calabi-Yau threefold, i.e., M admits a Ricci-flat Kähler metric.*

Corollary 3.12. *Let (\bar{X}, D) be an admissible pair with $\dim_{\mathbb{C}} \bar{X} = 3$. Then we can glue two copies of X along their cylindrical ends to obtain a compact manifold M . The manifold M is a Calabi-Yau threefold.*

Proof of Theorem 3.11. We shall prove the existence of a torsion-free G_2 -structure on $M_T \times S^1$ constructed in Section 3.3 for sufficiently large T . Then $M = M_T$ will be the desired Calabi-Yau threefold according to the following

Lemma 3.13. *If $M \times S^1$ admits a torsion-free G_2 -structure, then M admits a Ricci-flat Kähler metric.*

Proof. Since both X_1 and X_2 are simply-connected by Definition 3.6 of admissible pairs, the resulting manifold $M = M_T$ is also simply-connected. The S^1 -factor does not contribute to the holonomy, so that the holonomy group of M is contained in G_2 . But the holonomy group of simply-connected Riemannian 6-manifold is at most $\text{SO}(6)$, and so it must be contained in $\text{SO}(6) \cap G_2 = \text{SU}(3)$ and M admits a Ricci-flat Kähler metric. \square

Now it remains to prove the existence of a torsion-free G_2 -structure on $M_T \times S^1$ for sufficiently large T . This is a consequence of the following

Theorem 3.14 (Joyce [11], Theorem 11.6.1). *Let λ, μ, ν be positive constants. Then there exist positive constants ϵ_*, K such that whenever $0 < \epsilon < \epsilon_*$, the following is true.*

Let M be a compact 7-manifold, and (φ, g) a G_2 -structure on M with $d\varphi = 0$. Suppose ψ is a smooth 3-form on M with $d^\varphi = d^*\psi$, and*

- (1) $\|\psi\|_{L^2} \leq \lambda \epsilon^4$, $\|\psi\|_{C^0} \leq \lambda \epsilon^{1/2}$, and $\|d^*\psi\|_{L^{14}} \leq \lambda$,
- (2) *the injectivity radius $\delta(g)$ satisfies $\delta(g) \geq \mu \epsilon$, and*
- (3) *the Riemann curvature $R(g)$ satisfies $\|R(g)\|_{C^0} \leq \nu \epsilon^{-2}$.*

Then there exists a smooth, torsion-free G_2 -structure $(\tilde{\varphi}, \tilde{g})$ on M with $\|\tilde{\varphi} - \varphi\|_{C^0} \leq K \epsilon^{1/2}$.

Since X_1 and X_2 are cylindrical, the injectivity radius and Riemann curvature of M_T are uniformly bounded from below and above respectively, conditions (2) and (3) hold for sufficiently large T .

For condition (1), we define $\psi = \psi_T$ by equation (3.14) for $T > T_*$. Choosing γ so that $0 < \gamma < \min\{1/2, \sqrt{\lambda_1}\}/4$ and letting $\epsilon = e^{-\gamma T}$, we see from Proposition 3.10 that condition (1) holds for some λ . Thus we can apply Theorem 3.14 to prove that φ_T can be deformed into a torsion-free G_2 -structure for sufficiently large T . This completes the proof of Theorem 3.11. \square

4. BETTI NUMBERS OF THE RESULTING CALABI-YAU THREEFOLDS

We shall compute Betti numbers of the Calabi-Yau threefolds M obtained in the doubling construction given in Corollary 3.12. Also, we shall see that the Betti numbers of M is completely determined by those of the compact Kähler threefolds \bar{X} .

In our doubling construction, we take two copies (\bar{X}_j, D_j) of an admissible pair (\bar{X}, D) for $j = 1, 2$. Let $X_j = \bar{X}_j \setminus D_j$. We consider a homomorphism

$$(4.1) \quad \iota_j : H^2(X_j, \mathbb{R}) \longrightarrow H^2(D_j \times S^1, \mathbb{R}) \xrightarrow{\cong} H^2(D_j, \mathbb{R}),$$

where the first map is induced by the embedding $D_j \times S^1 \longrightarrow X_j$ and the second comes from the Künneth theorem. Set $d = d_j = \dim_{\mathbb{R}} \text{Ker } \iota_j$. It is readily seen that

$$(4.2) \quad \dim_{\mathbb{R}} \text{Im } \iota_j = b^2(X) - d.$$

Then we have the following

Proposition 4.1. *Let (\overline{X}_j, D_j) be two copies of an admissible pair (\overline{X}, D) for $j = 1, 2$ and let d be as above. Then the Calabi-Yau threefold obtained by the doubling construction in Corollary 3.12 has Betti numbers*

$$(4.3) \quad \begin{cases} b^1(M) = 0, \\ b^2(M) = b^2(\overline{X}) + d, \\ b^3(M) = 2(b^3(\overline{X}) + 23 + d - b^2(\overline{X})). \end{cases}$$

Also, the Euler characteristic $\chi(M)$ of M is given by

$$\chi(M) = 2(\chi(\overline{X}) - \chi(D)).$$

Proof. Obviously, the second statement holds for our construction. Now we restrict ourselves to find the second and third Betti numbers of M because M is simply-connected. Since the normal bundle N_{D_j/\overline{X}_j} is trivial in our assumption, there is a tubular neighborhood U_j of D_j in \overline{X}_j such that

$$(4.4) \quad \overline{X}_j = X_j \cup U_j \quad \text{and} \quad X_j \cap U_j \simeq D_j \times S^1 \times \mathbb{R}_{>0}.$$

Up to a homotopy equivalence, $X_j \cap U_j \sim D_j \times S^1$ as U_j contracts to D_j . Applying the Mayer-Vietoris theorem to (4.4), we see that

$$(4.5) \quad b^2(\overline{X}) = b^2(X) + 1 \quad \text{and} \quad b^3(X) = b^3(\overline{X}) + 22 + d - b^2(X)$$

(see [13], (2.10)). We next consider homotopy equivalences

$$(4.6) \quad M \sim X_1 \cup X_2, \quad X_1 \cap X_2 \sim D \times S^1.$$

Again, let us apply the Mayer-Vietoris theorem to (4.6). Then we obtain the long exact sequence

$$(4.7) \quad 0 \longrightarrow H^0(D) \xrightarrow{\delta_1} H^2(M) \xrightarrow{\alpha_2} H^2(X_1) \oplus H^2(X_2) \xrightarrow{\beta_2} H^2(D) \longrightarrow \cdots.$$

Note that the map β_2 in (4.7) is given by

$$\iota_1 + f^* \iota_2 : H^2(X_1, \mathbb{R}) \oplus H^2(X_2, \mathbb{R}) \longrightarrow H^2(D, \mathbb{R}),$$

where

$$\iota_j : H^2(X_j, \mathbb{R}) \longrightarrow H^2(D_j, \mathbb{R})$$

are homomorphisms defined in (4.1) and

$$f^* : H^2(D_2, \mathbb{R}) \longrightarrow H^2(D_1, \mathbb{R})$$

is the pull-back of the identity $f : D_1 \xrightarrow{\cong} D_2$. Hence we see from (4.2) that

$$\dim_{\mathbb{R}} \text{Im}(\iota_1 + f^* \iota_2) = b^2(X) - d.$$

This yields

$$\begin{aligned} b^2(M) &= \dim_{\mathbb{R}} \text{Ker } \alpha_2 + \dim_{\mathbb{R}} \text{Im } \alpha_2 \\ &= \dim_{\mathbb{R}} \text{Im } \delta_1 + \dim_{\mathbb{R}} \text{Ker}(\iota_1 + f^* \iota_2) \\ &= 1 + 2b^2(X) - (b^2(X) - d) = b^2(\overline{X}) + d, \end{aligned}$$

where we used (4.5) for the last equality. Remark that $b^2(X_1) = b^2(X_2)$ holds in our computation. To find $b^3(M)$, we shall consider a homomorphism

$$(4.8) \quad \tau_j : H^3(X_j, \mathbb{R}) \longrightarrow H^2(D_j, \mathbb{R})$$

which is induced by the embedding $U_j \cap X_j \longrightarrow X_j$ combined with

$$X_j \cap U_j \simeq D_j \times S^1 \times \mathbb{R}_{>0} \quad \text{and} \quad H^3(D_j \times S^1, \mathbb{R}) \cong H^2(D_j, \mathbb{R}).$$

The reader should be aware of the following lemma.

Lemma 4.2 (Kovalev-Lee [13]). *Let ι_j and τ_j be homomorphisms defined in (4.1) and (4.8) respectively. Then we have the orthogonal decomposition*

$$H^2(D_j, \mathbb{R}) = \text{Im } \tau_j \oplus \text{Im } \iota_j$$

with respect to the intersection form on $H^2(D_j, \mathbb{R})$ for each $j = 1, 2$.

Proof. See [13], Lemma 2.6. □

In an analogous way to the computation of $b^2(M)$, we apply the Mayer-Vietoris theorem to (4.6):

$$(4.9) \quad \begin{array}{ccccccc} \cdots & \longrightarrow & H^2(X_1) \oplus H^2(X_2) & \xrightarrow{\iota_1 + f^* \iota_2} & H^2(D) & \xrightarrow{\delta_2} & H^3(M) \longrightarrow \\ & & \xrightarrow{\alpha_3} & H^3(X_1) \oplus H^3(X_2) & \xrightarrow{\beta_3} & H^2(D) & \longrightarrow \cdots \end{array}$$

Similarly, the map β_3 is given by

$$\tau_1 + f^* \tau_2 : H^3(X_1) \oplus H^3(X_2) \longrightarrow H^2(D).$$

On one hand, Lemma 4.2 and (4.2) show that

$$\dim_{\mathbb{R}} \text{Im } \tau_j = 22 + d - b^2(X).$$

Hence we find that

$$(4.10) \quad \begin{aligned} \dim_{\mathbb{R}} \text{Ker}(\tau_1 + f^* \tau_2) &= b^3(X_1) + b^3(X_2) - \dim_{\mathbb{R}} \text{Im}(\tau_1 + f^* \tau_2) \\ &= 2b^3(X) - (22 + d - b^2(X)). \end{aligned}$$

On the other hand, we have the equality

$$22 = \dim_{\mathbb{R}} \text{Im } \delta_2 + \dim_{\mathbb{R}} \text{Im}(\iota_1 + f^* \iota_2)$$

by combining the well-known result on the cohomology of K3 surface D with the Mayer-Vietoris long exact sequence (4.9). Then we have

$$(4.11) \quad \dim_{\mathbb{R}} \text{Ker } \alpha_3 = \dim_{\mathbb{R}} \text{Im } \delta_2 = 22 - b^2(X) + d.$$

Thus we find from (4.10) and (4.11) that

$$b^3(M) = \dim_{\mathbb{R}} \text{Ker } \alpha_3 + \dim_{\mathbb{R}} \text{Ker}(\tau_1 + f^* \tau_2) = 2b^3(X).$$

Substituting the above equation into (4.5), we obtain the assertion. □

Remark 4.3. This formula shows that the topology of the resulting Calabi-Yau threefolds M only depends on the topology of the given compact Kähler threefolds \overline{X} . Also one can determine the Hodge diamond of M from Proposition 4.1 because we already know that $h^{0,0} = h^{3,0} = 1$ and $h^{1,0} = h^{2,0} = 0$ by the well-known result on Calabi-Yau manifolds (see [11], Proposition 6.2.6).

5. TWO TYPES OF ADMISSIBLE PAIRS

In this section, we will see the construction of admissible pairs (\overline{X}, D) which will be needed for obtaining Calabi-Yau threefolds in the doubling construction. There are two types of admissible pairs. One is said to be of *Fano type*, and the other of *non-symplectic type*. We will give explicit formulas for topological invariants of the resulting Calabi-Yau threefolds from these two types of admissible pairs. For the definition of admissible pairs, see Definition 3.6.

5.1. Fano type. Admissible pairs (\bar{X}, D) are ingredients in our construction of Calabi-Yau threefolds and then it is important how to explore appropriate compact Kähler threefolds \bar{X} with an anticanonical $K3$ divisor $D \in |-K_{\bar{X}}|$. In [12], Kovalev constructed such pairs from nonsingular Fano varieties.

Theorem 5.1 (Kovalev [12]). *Let V be a Fano threefold, $D \in |-K_V|$ a $K3$ surface, and let C be a smooth curve in D representing the self-intersection class of $D \cdot D$. Let $\varpi : \bar{X} \dashrightarrow V$ be the blow-up of V along the curve C . Taking the proper transform of D under the blow-up ϖ , we still denote it by D . Then (\bar{X}, D) is an admissible pair.*

Proof. See [12], Corollary 6.43, and also Proposition 6.42. \square

An admissible pair (\bar{X}, D) given in Theorem 5.1 is said to be of *Fano type* because this pair arises from a Fano threefold V . Note that \bar{X} itself is *not* a Fano threefold in this construction.

Proposition 5.2. *Let V be a Fano threefold and (\bar{X}, D) an admissible pair of Fano type given in Theorem 5.1. Let M be the Calabi-Yau threefold constructed from two copies of (\bar{X}, D) by Corollary 3.12. Then we have*

$$\begin{cases} b^2(M) = b^2(V) + 1, \\ b^3(M) = 2(b^3(V) - K_V^3 + 24 - b^2(V)). \end{cases}$$

In particular, the cohomology of M is completely determined by the cohomology of V .

Proof. Let d be the dimension of the kernel of the homomorphism

$$\iota : H^2(X, \mathbb{R}) \longrightarrow H^2(D, \mathbb{R})$$

as in Section 4. Then note that $d = 0$ by the Lefschetz hyperplane theorem whenever (\bar{X}, D) is of Fano type. From the standard results on the cohomology of blow-ups, one can find that

$$H^2(\bar{X}) \cong H^2(V) \oplus \mathbb{R} \quad \text{and} \quad H^3(\bar{X}) \cong H^3(V) \oplus \mathbb{R}^{2g(V)}$$

where $g(V) = \frac{-K_V^3}{2} + 1$ is the genus of a Fano threefold (see [12], (8.52)). This yields

$$b^2(\bar{X}) = b^2(V) + 1 \quad \text{and} \quad b^3(\bar{X}) = b^3(V) + 2g(V).$$

Substituting this into Proposition 4.1, we can show our result. \square

Remark 5.3. We have another method to compute the Euler characteristic $\chi(M)$. In fact, we can see easily that if \bar{X} is the blow-up of D along C then the Euler characteristic of \bar{X} is given by

$$\chi(\bar{X}) = \chi(V) - \chi(C) + \chi(E)$$

where E is the exceptional divisor of the blow-up ϖ . Hence we can independently compute $\chi(M)$ by

$$\begin{aligned} \chi(M) &= 2(\chi(\bar{X}) - \chi(D)) \\ &= 2(\chi(V) + \chi(C) - \chi(D)) \end{aligned}$$

because E is a $\mathbb{C}P^1$ -bundle over the smooth curve C . Since the Euler characteristic is also given by $\chi(M) = \sum_{i=0}^{\dim_{\mathbb{R}} M} (-1)^i b^i(M)$, we can check the consistency of our computations.

5.2. Non-symplectic type. In [13], Kovalev and Lee gave a large class of admissible pairs (\bar{X}, D) from $K3$ surface S with a non-symplectic involution ρ . They also used the classification result of $K3$ surfaces (S, ρ) due to Nikulin [16, 17, 18] for obtaining new examples of compact irreducible G_2 -manifolds. Next we will give a quick review on this construction. For more details, see [13], Section 4.

5.2.1. *K3 surfaces with a non-symplectic involution.* Let S be a $K3$ surface. Then the vector space $H^{2,0}(S)$ is spanned by a holomorphic volume form Ω , which is unique up to multiplication of a constant. An automorphism ρ of S is said to be *non-symplectic* if its action on $H^{2,0}(S)$ is nontrivial. We shall consider a non-symplectic involution:

$$\rho^2 = \text{id} \quad \text{and} \quad \rho^* \Omega = -\Omega.$$

The intersection form of S associates a lattice structure, i.e., a free abelian group of finite rank endowed with a nondegenerate integral bilinear form which is symmetric. We refer to this lattice as the *K3 lattice*. It is crucial that the $K3$ lattice has a nice property for a geometrical description of S . Hence we shall review some fundamental concepts of lattice theory which will be needed later.

Recall that the lattice L is said to be *hyperbolic* if the signature of L is $(1, t)$ with $t > 0$. In particular, we are interested in the case where L is *even*, i.e., the quadratic form x^2 is $2\mathbb{Z}$ -valued for any $x \in L$. We can regard L as a sublattice of $L^* = \text{Hom}(L, \mathbb{Z})$ by considering the canonical embedding $i : L \rightarrow L^*$ given by $i(x) = x \cdot y$ for $y \in L^*$. Then L is said to be *unimodular* if the quotient group L^*/L is trivial. In general, L^*/L is a finite abelian group and is called the *discriminant group* of L . One can see that the cohomology group $H^2(S, \mathbb{Z})$ of each $K3$ surface S is a unimodular, nondegenerate, even lattice with signature $(3, 19)$. Let H and E_8 denote the hyperbolic plane lattice $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and the root lattice associated to the root system E_8 respectively.

Then $H^2(S, \mathbb{Z})$ is isomorphic to $3H \oplus 2(-E_8)$. Let us choose a *marking* $\phi : H^2(S, \mathbb{Z}) \rightarrow L$ of S , that is, a lattice isomorphism. It is clear that the pull-back ρ^* induces an isometry of L with order 2 defined by $\phi \circ \rho^* \circ \phi^{-1}$. Hence we can consider the *invariant sublattice* L^ρ . Then L is said to be *2-elementary* if the discriminant group of L^ρ is isomorphic to $(\mathbb{Z}_2)^a$ for some $a \in \mathbb{Z}_{\geq 0}$.

Theorem 5.4 (Nikulin [16, 17, 18]). *Let (S, ρ) be a $K3$ surface S with a non-symplectic involution ρ . Then the deformation class of (S, ρ) depends only on the following triplet $(r, a, \delta) \in \mathbb{Z}^3$ given by*

- (i) $r = \text{rank } L^\rho$,
- (ii) $(L^\rho)^*/L^\rho \cong (\mathbb{Z}_2)^a$, and
- (iii) $\delta(L^\rho) = \begin{cases} 0 & \text{if } y^2 \in \mathbb{Z} \text{ for all } y \in L^\rho, \\ 1 & \text{otherwise.} \end{cases}$

5.2.2. *The cohomology for non-symplectic type.* Let σ be a holomorphic involution of \mathbb{CP}^1 given by

$$\sigma : \mathbb{CP}^1 \rightarrow \mathbb{CP}^1, \quad z \mapsto -z$$

in the standard local coordinates. Let G be the cyclic group of order 2 generated by $\rho \times \sigma$. Let X' be the trivial \mathbb{CP}^1 -bundle over S . Then the group G naturally acts on X' . Taking a point x in the fixed locus $W = (X')^G$ under the action of G , we denote the stabilizer of x as G_x . Then G_x is an endomorphism of the tangent space $T_x X'$ which satisfies $G_x \subset \text{SL}(T_x X')$. Define the quotient variety

$$Z = X'/G_x$$

and then the above condition $G_x \subset \text{SL}(T_x X')$ yields that the algebraic variety Z admits only Gorenstein quotient singularities [22]. Therefore, there is a crepant resolution $\bar{\pi} : \bar{X} \dashrightarrow Z$ due to Roan's result (see [19], Main theorem).

Now we review on the Chen-Ruan orbifold cohomology which will be needed for determining topological types of the resulting manifolds (see Proposition 5.9). Since $G_x \subset \text{SL}(T_x X')$ is a finite group, any element g in G_x has eigenvalues λ_i of the form $\lambda_i = \exp(2\pi a_i/r')$ with $0 \leq a_i < r'$ where r' is the order of g . Define the *age* of g at x by the rational number

$$\text{age}(g, x) = \frac{1}{r'} \sum_{i=1}^3 a_i \in \mathbb{Q}.$$

In particular, the property of Gorenstein quotient singularities of Z yields $\text{age}(g, x) \in \mathbb{Z}$ according to the main theorem in [22]. Let Σ be a connected component of the fixed point set $(X')^g = \{x \in X' \mid g(x) = x\}$. Then $\text{age}(g, x)$ is constant for all $x \in \Sigma$ and $g \in G_x$. Hence we denote this constant by $\text{age}(g, \Sigma)$. We note that $(X')^g$ coincides for all $g \neq \text{id}$. This gives the following Hodge decomposition of the orbifold cohomology of Z :

$$(5.1) \quad H_{\text{orb}}^{k, \ell}(Z) = H^{k, \ell}(X')^G \oplus \bigoplus_{\Sigma \subset (X')^\gamma} H^{k - \text{age}(\gamma, \Sigma), \ell - \text{age}(\gamma, \Sigma)}(\Sigma),$$

where $\gamma = \rho \times \sigma$ is the generator of $G \cong \mathbb{Z}_2$. In particular, we have the equalities

$$(5.2) \quad h^{k, \ell}(\overline{X}) = h_{\text{orb}}^{k, \ell}(\overline{X}) = h_{\text{orb}}^{k, \ell}(Z)$$

for all k and ℓ . For the first equality in (5.2), we used the fact that the orbifold cohomology coincides with the (original) Dolbeault cohomology when \overline{X} is a smooth variety [3]. The second equality in (5.2) follows from the following result by Yasuda.

Theorem 5.5 (Yasuda [23]). *Let V and V' be varieties with only Gorenstein quotient singularities. We assume that there are proper birational morphisms $\pi : Y \dashrightarrow V$ and $\pi' : Y \dashrightarrow V'$ with*

$$\pi^* K_V \cong \pi'^* K_{V'}.$$

Then V and V' have the same Hodge structure as orbifold cohomologies.

It is crucial that the topological type of \overline{X} does not depend on the choice of a crepant resolution by (5.2). Let W be the fixed locus of X' under the action of G as above. We assume that W is nonempty. In fact, this condition always holds unless $(r, a, \delta) = (10, 10, 0)$, i.e., S/ρ is an Enriques surface. Then it is known that W is the disjoint union of some rational curves. Let $\tilde{\pi} : \tilde{X} \dashrightarrow X'$ be the blow-up of $X' = S \times \mathbb{CP}^1$ along the fixed locus W . Then \tilde{X} is simply-connected as X' is simply-connected. Also, the action of G on X' lifts to the action of \tilde{G} on \tilde{X} as follows. Since we have the isomorphism

$$\tilde{X} \setminus \tilde{\pi}^{-1}(W) \cong X' \setminus W,$$

it suffices to consider the action of \tilde{G} on a point $x \in \tilde{\pi}^{-1}(W)$. Setting $g \cdot x = x$ for all $g \in \tilde{G}$ and $x \in \tilde{\pi}^{-1}(W)$, we have the lift \tilde{G} on \tilde{X} . Observe that $\tilde{X}/\tilde{G} \cong \overline{X}$ as the quotient of the variety \tilde{X} by \tilde{G} . Summing up these arguments, we have the following commutative diagram:

$$\begin{array}{ccccc} \tilde{G} & \xrightarrow{\text{lift}} & \tilde{X} & \xrightarrow{\tilde{f}} & \overline{X} \\ & & \downarrow \tilde{\pi} & & \downarrow \bar{\pi}: \text{crepant} \\ G & \curvearrowright & X' & \xrightarrow{f} & Z \end{array}$$

where \tilde{f} (resp. f) is the quotient map with respect to \tilde{G} (resp. G). Taking a non-fixed point $z \in \mathbb{CP}^1 \setminus \{0, \infty\}$, let us define $D' = S \times \{z\}$, which is a $K3$ divisor on X' . Setting D as the image of D' in Z , we still denote by D the proper transform of D' under $\bar{\pi}$. Then we can see that D is isomorphic to S . Furthermore, the normal bundle $N_{D/\overline{X}}$ is holomorphically trivial. In order to show (\overline{X}, D) is an admissible pair, we need the following lemmas.

Lemma 5.6 (Kovalev-Lee [13]). *\overline{X} is a compact Kähler threefold. Moreover, there exists a Kähler class $[\omega] \in H^2(\overline{X}, \mathbb{R})$ such that*

$$[\kappa] = [\omega|_D] \in H^2(D, \mathbb{R})$$

where $[\kappa]$ is a ρ -invariant Kähler class on D .

Proof. See [13], Proposition 4.1. □

Lemma 5.7 (Kovalev-Lee [13]). \overline{X} and $X = \overline{X} \setminus D$ are simply-connected whenever $(r, a, \delta) \neq (10, 10, 0)$.

Proof. See [13], Lemma 4.2. □

Lemma 5.8. D is an anticanonical divisor on \overline{X} .

Proof. To begin with, we consider the divisor $D' = S \times \{z\}$ on $X' = S \times \mathbb{C}P^1$, where $z \in \mathbb{C}P^1 \setminus \{0, \infty\}$. Let $p_1 : X' \rightarrow S$ and $p_2 : X' \rightarrow \mathbb{C}P^1$ be the canonical projections. Then we have the isomorphism

$$K_{X'} \cong p_1^* K_S \otimes p_2^* K_{\mathbb{C}P^1} \cong p_2^* \mathcal{O}_{\mathbb{C}P^1}(-2),$$

where we used $K_S \cong \mathcal{O}_S$ for the second isomorphism. Similarly, we conclude that

$$[D'] \cong p_2^*[z] \cong p_2^* \mathcal{O}_{\mathbb{C}P^1}(1).$$

This yields

$$K_{X'} \otimes [2D'] \cong \mathcal{O}_{X'}$$

and hence $c_1(K_{X'} \otimes [2D']) = 0$. Since $H^2(Z, \mathbb{Z})$ is the G -invariant part of $H^2(X', \mathbb{Z})$, the pullback map $f^* : H^2(Z, \mathbb{Z}) \rightarrow H^2(X', \mathbb{Z})$ is injective. Thus,

$$f^* c_1(K_Z \otimes [D]) = c_1(K_{X'} \otimes [2D']) = 0$$

implies $c_1(K_Z \otimes [D]) = 0$. We remark that

$$(5.3) \quad D \cap \text{Sing}(Z) = \emptyset$$

because $z \in \mathbb{C}P^1$ is a non-fixed point of σ . Since $\overline{\pi}$ is a crepant resolution, we have

$$\overline{\pi}^* K_Z \cong K_{\overline{X}} \quad \text{and} \quad \overline{\pi}^*[D] \cong [D]$$

by (5.3). Hence $c_1(K_Z \otimes [D]) = 0$ implies

$$c_1(K_{\overline{X}} \otimes [D]) = c_1(\overline{\pi}^* K_Z \otimes \overline{\pi}^*[D]) = \overline{\pi}^* c_1(K_Z \otimes [D]) = 0.$$

Now consider the long exact sequence

$$(5.4) \quad \cdots \longrightarrow H^1(\overline{X}, \mathcal{O}_{\overline{X}}) \longrightarrow H^1(\overline{X}, \mathcal{O}_{\overline{X}}^*) \xrightarrow{c_1} H^2(\overline{X}, \mathbb{Z}) \longrightarrow \cdots.$$

It follows from Lemmas 5.6 and 5.7 that $H^1(\overline{X}, \mathcal{O}_{\overline{X}}) \cong H^{0,1}(\overline{X}) = 0$. Thus the map c_1 in (5.4) is injective and so $c_1(K_{\overline{X}} \otimes [D]) = 0$ implies $K_{\overline{X}} \otimes [D] \cong \mathcal{O}_{\overline{X}}$. Hence D is an anticanonical divisor on \overline{X} . □

Therefore the above constructed pair (\overline{X}, D) is an admissible pair, which is said to be of *non-symplectic type* except the case of $(r, a, \delta) = (10, 10, 0)$.

Proposition 5.9. Let (S, ρ) be a K3 surface with a non-symplectic involution ρ which is determined by a K3 invariant (r, a, δ) up to a deformation. Let (\overline{X}, D) be the admissible pair of non-symplectic type obtained in the above construction from (S, ρ) . Let M denote the Calabi-Yau threefold constructed from two copies of (\overline{X}, D) by Corollary 3.12. Then the number of possibilities of the K3 invariants is 75. The number of topological types of (\overline{X}, D) which are distinguished by Betti or Hodge numbers is 64. Moreover, we have

$$\begin{cases} h^{1,1}(M) = b^2(M) = 5 + 3r - 2a, \\ h^{2,1}(M) = \frac{1}{2}b^3(M) - 1 = 65 - 3r - 2a. \end{cases}$$

Proof. Recall that we set $d = \dim_{\mathbb{R}} \text{Ker } \iota$, where

$$\iota : H^2(X, \mathbb{R}) \longrightarrow H^2(D, \mathbb{R})$$

is a homomorphism in (4.1). Let us define the restriction map given by

$$\iota' : H^2(\overline{X}, \mathbb{R}) \longrightarrow H^2(D, \mathbb{R}), \quad [\omega] \longmapsto [\omega|_D].$$

As in (4.3) in [13], we have

$$d = \dim_{\mathbb{R}} \text{Ker } \iota = \dim_{\mathbb{R}} \text{Ker } \iota' - 1.$$

Since $\dim_{\mathbb{R}} \text{Im } \iota' = r$ by Proposition 4.3 in [13], we conclude that

$$d = b^2(\overline{X}) - \dim_{\mathbb{R}} \text{Im } \iota' - 1 = h^{1,1}(\overline{X}) - r - 1.$$

Here we used the equality $h^{2,0}(\overline{X}) = 0$ given by Proposition 2.2 in [13]. Substituting this into (4.3) in Proposition 4.1, we have

$$(5.5) \quad \begin{cases} b^2(M) &= 2h^{1,1}(\overline{X}) - r - 1 \\ b^3(M) &= 2(2h^{2,1}(\overline{X}) + 22 - r). \end{cases}$$

In the above equation, we again used $h^{3,0}(\overline{X}) = 0$ by Proposition 2.2 in [13]. Hence we can reduce the problem to the computation of the Hodge diamond of the compact Kähler threefold \overline{X} . We can compute all Hodge numbers for possible compact Kähler threefolds \overline{X} by using the Chen-Ruan orbifold cohomology. Finally we will find the explicit form of the Betti numbers $b^i(M)$ in terms of $K3$ invariants.

We have already seen that the Hodge structure of \overline{X} is given in (5.1) and (5.2). To compute the right-hand side of (5.1), we note that

$$(5.6) \quad h^{*,*}(X') = \begin{array}{ccccc} & & 1 & & \\ & & 0 & & 0 \\ & 1 & & 21 & & 1 \\ 0 & 0 & & 0 & & 0 \\ & 1 & & 21 & & 1 \\ & & 0 & & 0 \\ & & & & 1 \end{array}$$

combining the known Hodge diamond of $K3$ surfaces and the Künneth theorem. We first compute the G -invariant part $H^{1,1}(X')^G$ in (5.6). Since the subspace $H^{1,1}(X')$ consists of $H^{1,1}(S)$ and $H^{1,1}(\mathbb{CP}^1)$, we shall compute $H^{1,1}(S)^\rho$ and $H^{1,1}(\mathbb{CP}^1)^\sigma$. Observe that

$$H^{1,1}(S, \mathbb{R})^\rho \cong L^\rho \otimes \mathbb{R}.$$

This yields $\dim_{\mathbb{R}} H^{1,1}(S, \mathbb{R})^\rho = r$. On the other hand, the subspace $H^{1,1}(\mathbb{CP}^1)$ is spanned by the cohomology class of the associated Fubini-Study Kähler form ω_{FS} on \mathbb{CP}^1 . This is σ -invariant. Thus we conclude that $h^{1,1}(X')^G = r + 1$. Similarly $H^{2,0}(X')$ is a one-dimensional subspace generated by a nowhere-vanishing holomorphic $(2, 0)$ -form on S . But obviously this is not ρ -invariant. Therefore we find that

$$h^{*,*}(X')^G = \begin{array}{ccccc} & & 1 & & \\ & & 0 & & 0 \\ & 0 & & r+1 & & 0 \\ 0 & 0 & & 0 & & 0 \\ & 0 & & r+1 & & 0 \\ & & 0 & & 0 \\ & & & & 1 \end{array}.$$

Suppose that $(r, a, \delta) \neq (10, 10, 0)$ and let γ be the generator of $G \cong \mathbb{Z}_2$. Then γ is given by

$$\gamma = \begin{pmatrix} -1 & & \\ & -1 & \\ & & 1 \end{pmatrix}$$

and the age of γ is 1. Next we shall consider the fixed locus $(X')^\gamma$. By the classification result due to Nikulin [16, 17, 18], the fixed locus S^ρ is the disjoint union of smooth curves of the following form:

$$S^\rho = \begin{cases} C_g \cup E_1 \cup \dots \cup E_k & \dots\dots\dots \text{(I)} \\ C^1 \cup C^2 & \dots\dots\dots \text{(II)} \end{cases}$$

where each E_i is a smooth rational curve (i.e., genus = 0) for $i = 1, \dots, k = \frac{1}{2}(r - a)$, C^j is a smooth elliptic curve for $j = 1, 2$, and C_g is a smooth curve of genus

$$g = \frac{1}{2}(22 - r - a).$$

In the case of the type (I), the equality

$$h^{*,*}(X')^\gamma = \begin{matrix} & 2k+2 & \\ & 2g & 2g \\ & 2k+2 & \end{matrix}$$

holds because the fixed locus $(X')^\gamma$ consists of two copies of S^ρ . Moreover, this equality also holds for the type (II) case. Then (5.2) and (5.1) give

$$h^{k,\ell}(\overline{X}) = h^{k,\ell}(X')^G + h^{k-1,\ell-1}(X')^\gamma.$$

Here we used $\text{age}(\gamma, \Sigma) = 1$ for any connected component Σ of $(X')^\gamma$. Thus we have

$$h^{*,*}(\overline{X}) = \begin{matrix} & & 1 & & & \\ & 0 & & 0 & & \\ & 0 & & 2r-a+3 & & 0 \\ h^{*,*}(\overline{X}) = & 0 & 22-r-a & & 22-r-a & 0 \\ & 0 & & 2r-a+3 & & 0 \\ & & 0 & & 0 & \\ & & & 1 & & \end{matrix}.$$

This result is consistent with Proposition 2.2 in [13]. Substituting the above equation into (5.5), we complete the proof. Remark that our result is independent of the integer δ . \square

6. APPENDIX: THE LIST OF THE RESULTING CALABI-YAU THREEFOLDS

In this section, we list all Calabi-Yau threefolds obtained in Corollary 3.12. We have the following two choices for constructing Calabi-Yau threefolds M :

- (a) Using an admissible pair (\bar{X}, D) of *Fano type*, one can determine the topological type of M by using the results of Mukai-Mori [14, 15]. According to the complete classification of nonsingular Fano threefolds V [7, 14, 15], there are 105 algebraic families with Picard number $1 \leq \rho(V) \leq 10$. Then the number of distinct topological types of the resulting Calabi-Yau threefolds is 59 (see Table 1 and fano.txt in Figure 1).
- (b) Using an admissible pair (\bar{X}, D) of *non-symplectic type*, one can find the topological types of M due to Nikulin [16, 17, 18]. According to the complete classification of $K3$ surfaces with non-symplectic involutions there are 74 algebraic families. Then the number of distinct topological types of the resulting Calabi-Yau threefolds is 64. Of these Calabi-Yau threefolds, there is at least one new example which is not diffeomorphic to the known ones (see Table 2, also see nonsymplectic.txt and newCY.txt in Figure 1).

6.1. All possible Calabi-Yau threefolds from Fano type. In Table 1, we hereby list the details of the resulting Calabi-Yau threefolds M from admissible pairs of Fano type. These topological invariants are computable by Proposition 5.2, and further details are left to the reader. In the table below, $\rho = \rho(V)$ denotes the Picard number of the Fano threefold V , and $h^{1,1} = h^{1,1}(M)$, $h^{2,1} = h^{2,1}(M)$ denote the Hodge numbers.

<i>Fano threefolds with $\rho = 1$</i>					<i>Fano threefolds with $\rho = 2$</i>				
No.	Label in [14]	$-K_V^3$	$h^{1,2}(V)$	$(h^{1,1}, h^{2,1})$	No.	Label in [14]	$-K_V^3$	$h^{1,2}(V)$	$(h^{1,1}, h^{2,1})$
1	—	2	52	(2, 128)	18	1	4	22	(3, 69)
2	—	4	30	(2, 86)	19	2	6	20	(3, 67)
3	—	6	20	(2, 68)	20	3	8	11	(3, 51)
4	—	8	14	(2, 58)	21	4	10	10	(3, 51)
5	—	10	10	(2, 52)	22	5	12	6	(3, 45)
6	—	12	7	(2, 48)	23	6	12	9	(3, 51)
7	—	14	5	(2, 46)	24	7	14	5	(3, 45)
8	—	16	3	(2, 44)	25	8	14	9	(3, 53)
9	—	18	2	(2, 44)	26	9	16	5	(3, 47)
10	—	22	0	(2, 44)	27	10	16	3	(3, 43)
11	—	8	21	(2, 72)	28	11	18	5	(3, 49)
12	—	16	10	(2, 58)	29	12	20	3	(3, 47)
13	—	24	5	(2, 56)	30	13	20	2	(3, 45)
14	—	32	2	(2, 58)	31	14	20	1	(3, 43)
15	—	40	0	(2, 62)	32	15	22	4	(3, 51)
16	—	54	0	(2, 76)	33	16	22	2	(3, 47)
17	—	64	0	(2, 86)	34	17	24	1	(3, 47)
					35	18	24	2	(3, 49)
					36	19	26	2	(3, 51)
					37	20	26	0	(3, 47)
					38	21	28	0	(3, 49)
					39	22	30	0	(3, 51)

No.	Label in [14]	$-K_V^3$	$h^{1,2}(V)$	$(h^{1,1}, h^{2,1})$
40	23	30	1	(3, 53)
41	24	30	0	(3, 51)
42	25	32	1	(3, 55)
43	26	34	0	(3, 55)
44	27	38	0	(3, 59)
45	28	40	1	(3, 63)
46	29	40	0	(3, 61)
47	30	46	0	(3, 67)
48	31	46	0	(3, 67)
49	32	48	0	(3, 69)
50	33	54	0	(3, 75)
51	34	54	0	(3, 75)
52	35	56	0	(3, 77)
53	36	62	0	(3, 83)

Fano threefolds with $\rho = 3$

No.	Label in [14]	$-K_V^3$	$h^{1,2}(V)$	$(h^{1,1}, h^{2,1})$
54	1	12	8	(4, 48)
55	2	14	3	(4, 40)
56	3	18	3	(4, 44)
57	4	18	2	(4, 42)
58	5	20	0	(4, 40)
59	6	22	1	(4, 44)
60	7	24	1	(4, 46)
61	8	24	0	(4, 44)
62	9	26	3	(4, 52)
63	10	26	0	(4, 46)
64	11	28	1	(4, 50)
65	12	28	0	(4, 48)
66	13	30	0	(4, 50)
67	14	32	1	(4, 54)
68	15	32	0	(4, 52)
69	16	34	0	(4, 54)
70	17	36	0	(4, 56)
71	18	36	0	(4, 56)
72	19	38	0	(4, 58)
73	20	38	0	(4, 58)
74	21	38	0	(4, 58)
75	22	40	0	(4, 60)
76	23	42	0	(4, 62)
77	24	42	0	(4, 62)
78	25	44	0	(4, 64)
79	26	46	0	(4, 66)
80	27	48	0	(4, 68)
81	28	48	0	(4, 68)
82	29	50	0	(4, 70)
83	30	50	0	(4, 70)
84	31	52	0	(4, 72)

Fano threefolds with $\rho = 4$

No.	Label in [14]	$-K_V^3$	$h^{1,2}(V)$	$(h^{1,1}, h^{2,1})$
85	1	24	1	(5, 45)
86	2	28	1	(5, 49)
87	3	30	0	(5, 49)
88	4	32	0	(5, 51)
89	5	32	0	(5, 51)
90	6	34	0	(5, 53)
91	7	36	0	(5, 55)
92	8	38	0	(5, 57)
93	9	40	0	(5, 59)
94	10	42	0	(5, 61)
95	11	44	0	(5, 63)
96	12	46	0	(5, 65)
97*	—	26	0	(5, 45)

*) No. 97 was erroneously omitted in [14].
See [15] for the correct table.

Fano threefolds with $\rho \geq 5$

No.	ρ	$-K_V^3$	$h^{1,2}(V)$	$(h^{1,1}, h^{2,1})$
98	5	28	0	(6, 46)
99	5	36	0	(6, 54)
100 [†]	5	36	0	(6, 54)
101	6	30	0	(7, 47)
102	7	24	0	(8, 40)
103	8	18	0	(9, 33)
104	9	12	0	(10, 26)
105	10	6	0	(11, 19)

[†]) This Fano threefold is $\mathbb{C}P^1 \times S_6$ where S_6 is a del Pezzo surface of degree 6.

Table 1: The list of Calabi-Yau threefolds from Fano type.

6.2. All possible Calabi-Yau threefolds from non-symplectic type. In Table 2, we hereby list the details of the resulting Calabi-Yau threefolds from admissible pairs of non-symplectic type. These Hodge numbers are also computable by Proposition 5.9 and further details are left to the reader. In the table below, there is at least *one* new example of Calabi-Yau threefolds, which is listed as the boxed number 64. We also list the number of the mirror partner for each resulting Calabi-Yau threefold in our construction. See *Discussion* and Section 6.3 below for more details. The symbol – on the list means that the corresponding Calabi-Yau threefold has no mirror partner in this construction.

K3 surfaces with non-symplectic involutions

No.	$K3$ invariants (r, a, δ)	$(h^{1,1}, h^{2,1})$	Mirror partner	No.	$K3$ invariants (r, a, δ)	$(h^{1,1}, h^{2,1})$	Mirror partner
1	(2, 0, 0)	(11, 59)	3	36	(18, 4, 0 or 1)	(51, 3)	–
2	(10, 0, 0)	(35, 35)	2	37	(5, 5, 1)	(10, 40)	42
3	(18, 0, 0)	(59, 11)	1	38	(7, 5, 1)	(16, 34)	41
4	(1, 1, 1)	(6, 60)	9	39	(9, 5, 1)	(22, 28)	40
5	(3, 1, 1)	(12, 54)	8	40	(11, 5, 1)	(28, 22)	39
6	(9, 1, 1)	(30, 36)	7	41	(13, 5, 1)	(34, 16)	38
7	(11, 1, 1)	(36, 30)	6	42	(15, 5, 1)	(40, 10)	37
8	(17, 1, 1)	(54, 12)	5	43	(17, 5, 1)	(46, 4)	–
9	(19, 1, 1)	(60, 6)	4	44	(6, 6, 1)	(11, 35)	48
10	(2, 2, 0 or 1)	(7, 55)	18	45	(8, 6, 1)	(17, 29)	47
11	(4, 2, 1)	(13, 49)	17	46	(10, 6, 0 or 1)	(23, 23)	46
12	(6, 2, 0)	(19, 43)	16	47	(12, 6, 1)	(29, 17)	45
13	(8, 2, 0)	(25, 37)	15	48	(14, 6, 0 or 1)	(35, 11)	44
14	(10, 2, 0 or 1)	(31, 31)	14	49	(16, 6, 1)	(41, 5)	–
15	(12, 2, 1)	(37, 25)	13	50	(7, 7, 1)	(12, 30)	53
16	(14, 2, 0)	(43, 19)	12	51	(9, 7, 1)	(18, 24)	52
17	(16, 2, 1)	(49, 13)	11	52	(11, 7, 1)	(24, 18)	51
18	(18, 2, 0 or 1)	(55, 7)	10	53	(13, 7, 1)	(30, 12)	50
19	(20, 2, 1)	(61, 1)	–	54	(15, 7, 1)	(36, 6)	–
20	(3, 3, 1)	(8, 50)	27	55	(8, 8, 1)	(13, 25)	57
21	(5, 3, 1)	(14, 44)	26	56	(10, 8, 0 or 1)	(19, 19)	56
22	(7, 3, 1)	(20, 38)	25	57	(12, 8, 1)	(25, 13)	55
23	(9, 3, 1)	(26, 32)	24	58	(14, 8, 1)	(31, 7)	–
24	(11, 3, 1)	(32, 26)	23	59	(9, 9, 1)	(14, 20)	60
25	(13, 3, 1)	(38, 20)	22	60	(11, 9, 1)	(20, 14)	59
26	(15, 3, 1)	(44, 14)	21	61	(13, 9, 1)	(26, 8)	–
27	(17, 3, 1)	(50, 8)	20	62	(10, 10, 1) [‡]	(15, 15)	62
28	(19, 3, 1)	(56, 2)	–	63	(12, 10, 1)	(21, 9)	–
29	(4, 4, 1)	(9, 45)	35	64	(11, 11, 1)	(16, 10)	–
30	(6, 4, 0 or 1)	(15, 39)	34				
31	(8, 4, 1)	(21, 33)	33				
32	(10, 4, 0 or 1)	(27, 27)	32				
33	(12, 4, 1)	(33, 21)	31				
34	(14, 4, 0 or 1)	(39, 15)	30				
35	(16, 4, 1)	(45, 9)	29				

[‡] (r, a, δ) \neq (10, 10, 0) from assumption.

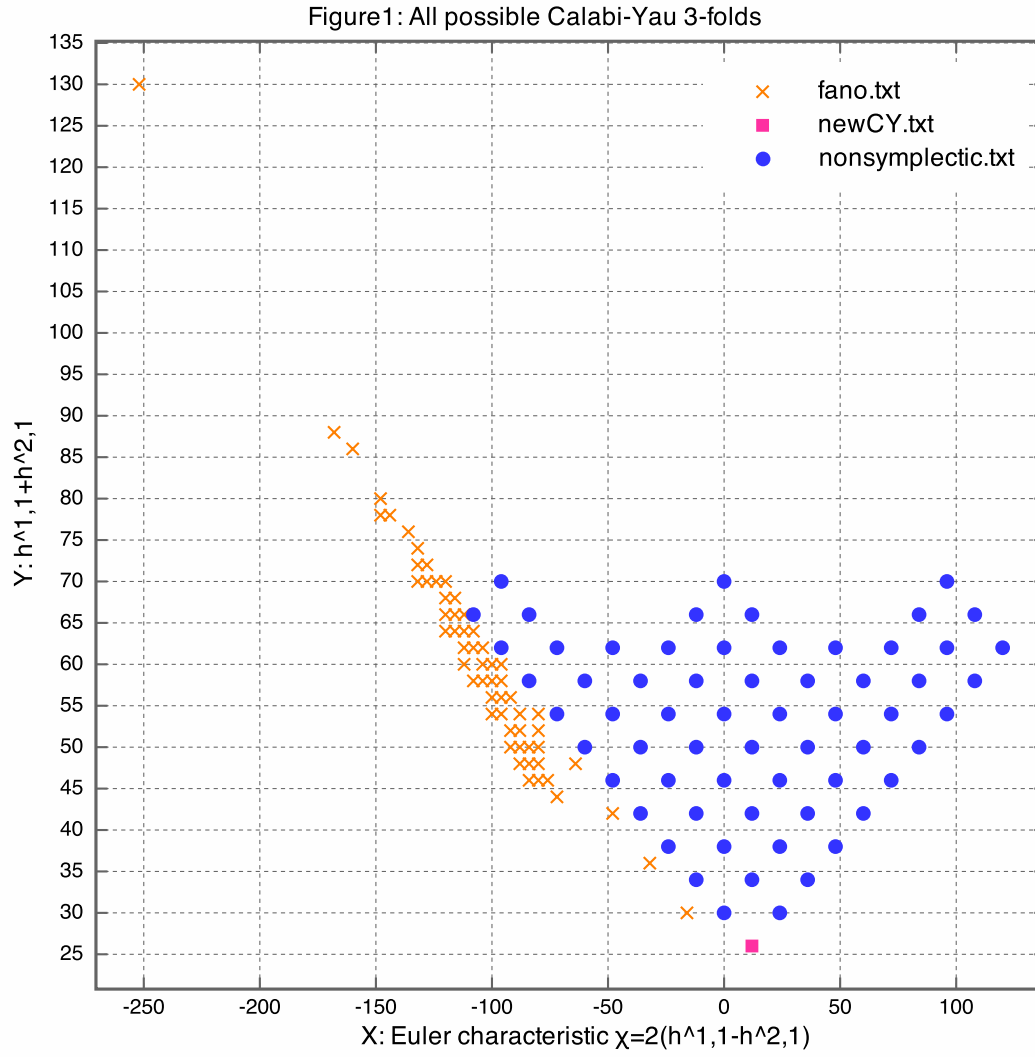
Table 2: The list of Calabi-Yau threefolds from non-symplectic type.

Discussion. Observe that Proposition 5.9 gives the condition that two Calabi-Yau threefolds M and M' should be a mirror pair, i.e., $h^{p,q}(M) = h^{3-p,q}(M')$ for all $p, q \in \{0, 1, 2, 3\}$. Let M (resp. M') be a Calabi-Yau threefold from admissible pairs of non-symplectic type with respect to $K3$ invariants (r, a, δ) (resp. (r', a', δ')). Then $h^{p,q}(M) = h^{3-p,q}(M')$ implies $r + r' = 20$, $a = a'$ by Proposition 5.9. From these relations, we can find mirror pairs in our examples of Calabi-Yau threefolds. In particular M is automatically self-mirror when $r = 10$. Thus we find 24 mirror pairs and 6 self-mirror Calabi-Yau threefolds in our examples.

6.3. Graphical chart of our examples. Finally we plot the Hodge numbers of the resulting Calabi-Yau threefolds (Figure 1). In the following figure, Calabi-Yau threefolds obtained from Fano type (Type (a) case) are registered as fano.txt and those from non-symplectic type (Type (b) case) are registered as nonsymplectic.txt. Separately, our new example is denoted by solid square (newCY.txt) in Figure 1. We take the Euler characteristic $\chi = 2(h^{1,1} - h^{2,1})$ along the X -axis and $h^{1,1} + h^{2,1}$ along the Y -axis. We see that all our examples from non-symplectic type are located on the integral lattice of the form

$$(6.1) \quad (X, Y) = (12, 26) + m(12, 4) + n(-12, 4), \quad m, n \in \mathbb{Z}_{\geq 0}.$$

In this plot the mirror symmetry is considered as the inversion $\mu : (X, Y) \mapsto (-X, Y)$ with respect to the Y -axis. The set of 56 points with $n > 0$ in (6.1) is μ -invariant, and thus the corresponding Calabi-Yau threefolds have a mirror partner in this set.



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